Forced and Free Convection Turbulent Boundary Layers in Gas Lasers

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Introduction

THERE is interest in high-power lasers which operate with a slowly flowing gas at temperatures below room temperature,[1] while the temperature level of the cavity is that of the surrounding environment. Medium homogeneity requirements for a diffraction-limited output beam place stringent limits on the allowable variation in optical path length in the cavity. The density variations in the warm fluid mechanic boundary layers on the solid windows can be major contributors to this total variation. Since the windows are warmer than the gas, buoyancy effects appear along with forced convection in the boundary layer. The optimum design is one where the flow is vertical and directed upward to ensure that buoyancy effects are favorable. The boundary-layer flow may be laminar or turbulent for operating conditions of interest. This Note presents a technique for calculating the optical effects of combined forced and free convection turbulent boundary layers.

The change in optical path length $\Delta\ell$ due to the variation in gas density in the boundary layer is, relative to uniform medium,

$$\Delta\ell = \int_0^{\delta} \frac{\rho - \rho_\infty}{\rho_\infty}\, dy \tag{1}$$

where δ is the boundary-layer thickness, ρ is density in the boundary layer, ρ_∞ is the gas density far away from the window and y is distance perpendicular to the window. The change in path length which is caused by turbulent fluctuations in the boundary layer is much smaller than $\Delta\ell$ in Eq. (1), and will be ignored. If $\Delta\ell$ did not vary from point to point (in the stream-wise direction) in the boundary layer, each ray would travel the same path length. It is the change in $\Delta\ell$ from point to point as the boundar layer has been obtained analytically and numerically by several authors.[2-6] We are interested here in the turbulent case. An approximate solution for the first order perturbation on the asymptotic solutions for pure forced convection and pure free convection using an integral formulation for the boundary-layer equations will be obtained. Characteristic length scales for optical path variations will be presented.

Forced Convection Asymptotic Case

In the absence of pressure gradients and body forces the boundary-layer momentum equation for the pure forced convection case is,

$$\rho \int_0^{\delta_l} u(u_\infty - u)\, dy = \tau_w \tag{2}$$

where δ_l is the pure forced convection boundary-layer thickness, u is velocity, u_∞ is velocity far away from the wall, and τ_w is wall shear stress.

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Assuming

$$u = u_\infty (y/\delta_l)^{1/7} \tag{3}$$

$$\tau_w = 0.0228\, \rho u_\infty^2 (\nu / u_\infty \delta_l)^{1/4}$$

the well known result for δ_l is obtained,

$$\delta_l = 0.376\, (\nu/u_\infty)^{1/5}\, x^{4/5} = 0.376\, x\, (Re_x)^{1/5} \tag{4}$$

where ν is the kinematic viscosity, x is distance along the window, and Re_x is Reynolds number, $Re_x = u_\infty x/\nu$.

If the boundary layer is heated in a gravity field, buoyancy forces arise. If the buoyancy force is included in the momentum equation using the approximation $\Delta\rho/\rho = (T - T_\infty)/T_\infty$, Eq. (2) becomes

$$\rho \int_0^{\delta_l + \delta_l'} u(u_\infty - u)\, dy = \tau_w - (\rho g / T_\infty) \int_0^{\delta_l + \delta_l'} \Theta dy \tag{5}$$

where g is the acceleration of gravity, T_∞ is the gas temperature far away from the wall, $\Theta = T - T_\infty$, and δ_l' is the perturbation in boundary-layer thickness caused by the addition of buoyancy. We will take

$$\Theta = \Theta_w [1 - (y/\delta)^{1/7}] \tag{6}$$

(where Θ_w is $T_{\text{wall}} - T_\infty$). We will also asume that the buoyancy force is small, and that $\delta_l' \ll \delta_l$. It is possible to derive equations for parameters describing the perturbations in boundary-layer velocity and temperature from the momentum and energy equations taking constant δ_l. This procedure requires that one choose suitable profiles for the perturbations. Since there is no information available on such profiles, any profile choice would necessarily be arbitrary. For this reason, we will assume that buoyancy appears as a change δ_l' in the boundary-layer thickness with the unperturbed velocity profile [Eq. (3)]. While this procedure cannot be expected to give the correct numerical value for the perturbation, the functional dependence obtained should be the correct one. The numerical constants in the perturbation results should be regarded as being accurate only to within some small factor. Then we can obtain an equation for the perturbation in δ_l caused by buoyancy, by subtracting Eq. (2) from Eq. (5), and using Eqs. (3, 4, and 6),

$$\frac{7}{72} u_\infty^2 \frac{d\delta_l'}{dx} = - \frac{0.0228}{4(0.376)^{5/4}} \frac{u_\infty^2}{x} \delta_l' - \frac{0.376}{8} g \frac{\Theta_w}{T_\infty} \left(\frac{\nu}{u_\infty}\right)^{1/5} x^{4/5} \tag{7}$$

Trying a solution $\delta_l' \sim Cx^n$ leads to the result

$$\delta_l' \approx -0.24 (\nu/u_\infty)^{1/5} (g(\Theta_w/T_\infty)/u_\infty^2) x^{9/5} \tag{8}$$

Buoyancy thus reduces the boundary-layer thickness at a given value of x, since the perturbed thickness is $(\delta_l + \delta_l')$.

Free-Convection Asymptotic Case

Eckert and Jackson[8] have solved the problem of a pure free-convection turbulent boundary layer. Taking integral equations for momentum and energy

$$\rho (d/dx) \int_0^{\delta_2} u^2\, dy = (\rho g / T_\infty) \int_0^{\delta_2} \Theta\, dy - 0.0228\, \rho u_2^2\, (\nu/u_2 \delta)^{1/4} \tag{9}$$

$$\rho C_p (d/dx) \int_0^{\delta_2} u\, \Theta\, dy = q_w = 0.0228\, \rho C_p u_2 \Theta_w (\nu/u_2 \delta)^{1/4} \tag{10}$$

where δ_2 is the pure free convection layer thickness, C_p is the gas specific heat at constant pressure and q_w is the wall heat transfer, and using

$$u=u_2(y/\delta)^{1/7}(1-(y/\delta))^4 \tag{11}$$

$$\Theta=\Theta_w[1-(y/\delta)^{1/7}] \tag{12}$$

then following Eckert and Jackson for $P_r=1$, one can obtain

$$\delta_2=0.595\ (Gr_x)^{-1/10}x \tag{13}$$

where Gr_x is the Grashof number, $Gr_x=g(\Theta_w/T_\infty)x^3/\nu^2$. For comparison, in the laminar case $\delta_2 \sim x^{1/4}$, instead of $\delta_2 \sim x^{7/10}$ as in Eq. (13).

Introducting a perturbation consisting of an external flow (velocity u_∞) into the energy equation, Eq. (10), we obtain

$$\rho\, C_p(\mathrm{d}/\mathrm{d}x)\int_0^{\delta_2+\delta_2'} (u+u_\infty)\ \Theta \mathrm{d}y=q_w \tag{14}$$

Proceeding as before, we obtain an equation for the perturbation in δ_2 caused by the external flow,

$$0.0366\ (\mathrm{d}/\mathrm{d}x)\ (C_1x^{1/2}\delta_2')+(7/10)u_\infty\ C_2x^{-3/10}/8$$
$$=(0.0228)\,3/4\,(u_\infty\nu^{1/4}/C_1{}^{1/4}C_2{}^{1/4})x^{-3/10}$$
$$-0.0228\ (1/4)\nu^{1/4}(C_1{}^{3/4}/C_2{}^{5/4})x^{-1/2}\delta_2' \tag{15}$$

where δ_2' is the perturbation in δ_2,

$$C_1=0.972\ [g(\Theta_w/T_\infty)/\nu^2]^{1/2}$$

and

$$C_2=0.595\ [g(\Theta_w/T_\infty)/\nu^2]^{-1/10}$$

Again trying a solution of form $\delta_2' \sim Cx^n$ leads to the result

$$\delta_2' \approx -0.91(u_\infty/\nu)\ [g(\Theta_w/T_\infty)/\nu^2]^{-6/10}x^{2/10} \tag{16}$$

The external flow reduces δ_2 at a given x, since the perturbed thickness is $(\delta_2+\delta_2')$.

Results for Optical Path

We now want to use the results just obtained for the perturbed boundary-layer thickness to calculate optical path lengths through the boundary layer,

Using the approximation $(\rho-\rho_\infty)/\rho_\infty=\Theta/T_\infty$, Eq. (1) becomes

$$\Delta\ell=\int_0^\delta (\Theta/T_\infty)\ dy \tag{17}$$

Using Eqs. (6) for Θ and Eqs. (4) and (8) for δ in Eq. (17), we obtain for the forced-convection asymptotic case

$$\Delta\ell_1 \approx 0.047\ (\Theta_w/T_\infty)x\,Re_x{}^{-1/5}[1-0.6\ (Gr_x/Re_x{}^2)] \tag{18}$$

Similarly, using Eqs. (12, 13, and 16)

$$\Delta\ell_2 \approx 0.074\ (\Theta_w/T_\infty)x\ Gr_x{}^{-1/10}\ [1-1.5\ (Re_x/Gr_x{}^{1/2})] \tag{19}$$

The dominant parameter in this problem is a Froude number based on x, where

$$Fr_x=\frac{u_\infty}{[g(\Theta_w/T_\infty)x]^{1/2}}=\frac{Re_x}{(Gr_x)^{1/2}} \tag{20}$$

There is then a scale length L^* for distance along the window,

$$L^*=u_\infty{}^2/g(\Theta_w/T_\infty) \tag{21}$$

If $x \ll L^*$, forced convection is dominant. As x becomes very large $(x \gg L^*)$, free convection (i.e., buoyancy) dominates. Neither Eq. (18) or (19) applies when $x/L^* \sim 1$. In nondimensional form, these results are

$$\Delta\ell_1{}^* \approx 0.047(x^*)^{4/5}(1-0.6x^*) \quad \text{(Forced Convection)} \tag{22}$$

$$\Delta\ell_2{}^* \approx 0.074(x^*)^{7/10}(1-1.5(x^*)^{-1/2}) \quad \text{(Free Convection)} \tag{23}$$

where

$$x^*=x/L^* \tag{24}$$

$$\Delta\ell^*=\Delta\ell/Y^*_{\mathrm{TURB}} \tag{25}$$

where Y^*_{TURB} is a scaling length for the optical path variation, and is defined by

$$Y^*_{\mathrm{TURB}}=[(\Theta_w/T_\infty)^{1/5}u_\infty{}^{7/5}\nu^{1/5}/g^{4/5}] \tag{26}$$

As stated before Eq. (7) the numerical values 0.6 and 1.5 in Eqs. (22) and (23) should be regarded as accurate only to within some small factor.

Following a procedure similar to that presented for a laminar layer leads to a different scaling length Y^*_{LAM},

$$Y^*_{\mathrm{LAM}}=\left[\frac{(\Theta_w/T_\infty)u_\infty\nu}{g}\right]^{1/2} \tag{27}$$

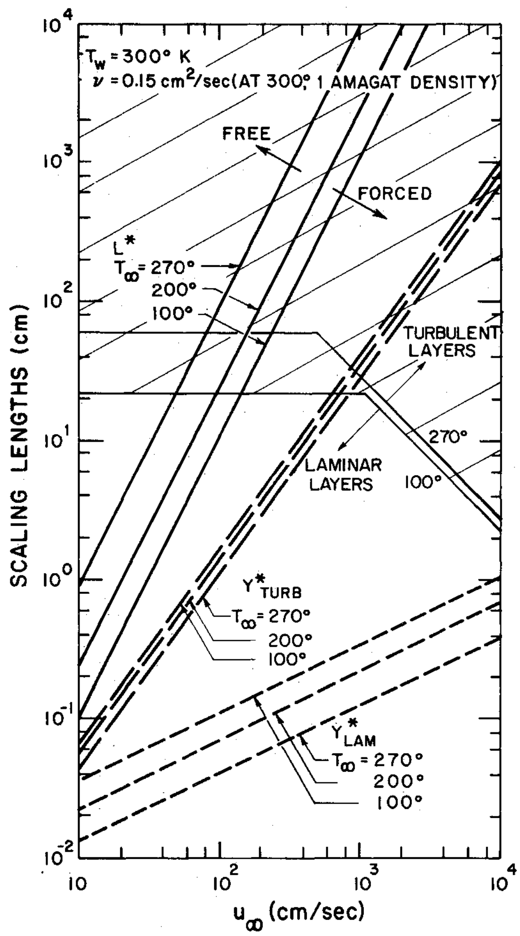

Fig. 1 Scaling lengths vs gas flow velocity u_∞ and gas temperature T_∞ for a fixed window temperature, 300° K, for CO-N_2 mixtures (any mixture ratio). For a given T_∞, downstream distances below and gas velocities to the right of the L^* line for that T_∞ correspond to forced-convection dominated flow. The region above and to the left corresponds to convection dominated flow The region in x and u_∞ where the flow is turbulent at T_∞ = 100°K is shown shaded.

while L^* is as in Eq. (21). Values for L^*, Y^*_{TURB} and Y^*_{LAM} as functions of u_∞ and Θ_w/T_∞ for CO, N_2 mixtures at 1 amagat density are shown in Fig. 1. Since the values of ν for CO and N_2 are nearly identical, the mixture ratio is unimportant. In Fig. 1, (Θ_w/T_∞) and ν have been evaluated using a reference temperature halfway between T_w and T_∞ to replace T_∞ for large $(T_w - T_\infty)$. Regions in x, u_∞, and T_∞ where the flow is turbulent have been delineated using the criteria $Re_x = 2 \times 10^5$ or $Gr_x = 10^9$.

Conclusion

Approximate expressions for the effect on optical path length through a turbulent vertical boundary layer caused by the combined presence of forced and free convection have been obtained to first order in the asymptotic cases of dominant forced convection and dominant free convection. The effect in both cases is a reduction of the boundary-layer thickness. Characteristic scaling lengths have been presented which aid in the optical analysis of the flowfield.

References

[1] Center, R. E., "High Pressure Electrical CO Laser," *Journal of Quantum Electronics*, Vol. QE-10, Oct. 1974, pp. 208-213.

[2] Szewczyk, A. A., "Combined Forced and Free Convection Laminar Flow," *Transactions of the ASME, Journal of Heat Transfer*, Vol. 86, Nov. 1964, pp. 501-507.

[3] Merkin, J. H., "The Effect of Buoyancy Forces on the Boundary Layer Flow over a Semi-Infinite Vertical Flat Plate in a Uniform Free Stream," *Journal of Fluid Mechanics*, Vol. 35, Pt. 3, 1969, pp. 439-450.

[4] Lloyd, J. R. and Sparrow, E. M., "Combined Forced and Free Convection Flow on Vertical Surfaces," *International Journal of Heat and Mass Transfer*, Vol. 13, Feb. 1970, pp. 434-438.

[5] Oosthuizen, P. H. and Hart, R., "A Numerical Study of Laminar Combined Convection Flow over Flat Plates," *Transactions of the ASME, Journal of Heat Transfer*, Vol. 95, Feb. 1973, pp. 60-63.

[6] Wilks, G., "Combined Forced and Free Convection Flow on Vertical Surfaces," *International Journal of Heat and Mass Transfer*, Vol. 16, Oct. 1973, pp. 1958-1963.

[7] Eckert, E.R.G., *Heat and Mass Transfer*, McGraw-Hill, New York, 1959.

[8] Eckert, E. R. G. and Jackson, T. W., "Analysis of Turbulent Free-Convection Boundary Layer on Flat Plate," Rept. 1015, 1951, NACA.

Cathode Geometry and Spoke Mode Operation of MPD Accelerators

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Introduction

SEVERAL authors have reported azimuthal nonuniformities, or rotating spokes, in the arc region and exhaust plume of MPD accelerators.[1-6] The amplitude and frequency of these rotating disturbances have been shown to be dependent on the propellant gas molecular weight, mass flow rate, applied current, and applied magnetic field strength.[1,2,5] Larson[3] and Allario, Jarrett and Hess[4] have shown the presence of a critical magnetic field strength at which transition from uniform to spoke-like discharge occurs. In addition, several investigators have attempted to provide theoretical bases for the observed spoke phenomena.[7-9]

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All of these studies have presented correlations of spoke mode operation with the accelerator operating parameters. As Cochran and Fay[5] have pointed out, no systematic study of the effect of electrode geometry on the occurence of the spoke mode has been made to date. However, this Note presents experimental results which suggest that the electrode geometry may have a significant effect on the mode of accelerator operation.

Experiment

An investigation of possible spoke mode operation was carried out for the plasma accelerator configuration shown in Fig. 1. The device was geometrically similar to that used by Larson[3] except for the cathode design, which was modified to provide increased cathode spot stability. The cathode was made from 0.64 cm o.d. thoriated tungsten rod, with a 0.32 cm i.d. hole, approximately 1.0 cm deep, drilled in the tip. This cavity provided stable cathode attachment for all operating conditions encountered in this investigation. In contrast, when a cone-tipped cathode was used in the otherwise identical accelerator the cathode attachment point moved erratically over the tip surface for applied magnetic field strengths below 0.05 tesla.

The effect of approximately 40 hours running time on the simple hollow cathode of this study is illustrated in Fig. 2. The erosion pattern indicates arc attachment within the cavity.

Since the luminosity of the rotating spoke has been shown to be significantly different from the rest of the arc region,[6] an optical technique was used to identify spoke mode operation. An image of a small portion of the arc region plasma located at a radial position midway between the anode

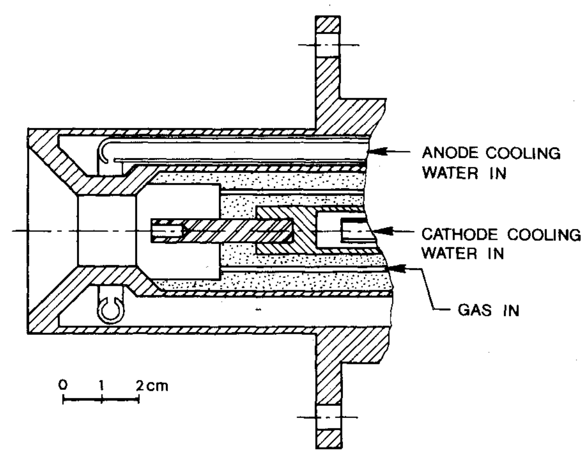

Fig. 1 Schematic cross-section of plasma accelerator.

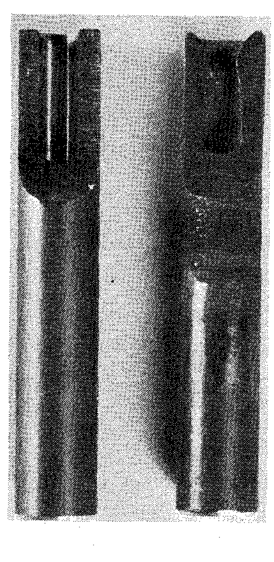

Fig. 2 Section view of new and used cathode inserts.